Some Geometric Properties of a New Modular Space Defined by Zweier Operator


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Some geometric properties of a new modular space defined by Zweier operator

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Dedicated to Professor Hari M Srivastava

Abstract. In this paper, we define the modular space $Z_0(x, p)$ by using the Zweier operator and a modular. Then, we consider it equipped with the Luxemburg norm and also examine the uniform Opial property and property $\beta$. Finally, we show that this space has the fixed point property.

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1. Introduction

In literature, there are many papers about the geometrical properties of different sequence spaces such as [5], [6], [11], [12], [13], [15], [17], [19], [20]. Opial [18] introduced the Opial property and proved that the sequence spaces $\ell_p$ $(1 < p < \infty)$ have this property but $L_p[0, 2\pi]$ $(p \neq 2, 1 < p < \infty)$ does not have it. Franchetti [7] showed that any infinite dimensional Banach space has an equivalent norm that satisfies the Opial property. Later, Prus [21] introduced and investigated the uniform Opial property for Banach spaces. The Opial property is important because Banach spaces with this property have the weak fixed point property.

2. Definition and preliminaries

Let $(X, \|\|)$ be a real Banach space and let $S(X)$ (resp. $B(X)$) be the unit sphere (resp. the unit ball) of $X$. A Banach space $X$ has the Opial property if for any weakly null sequence $\{x_n\}$ in $X$ and any $x$ in $X/\{0\}$, the inequality $\liminf_{n \to \infty} \|x_n + x\| < \liminf_{n \to \infty} \|x_n - x\|$ holds. We say that $X$ has the uniform Opial property if for any $\varepsilon > 0$ there exists $r > 0$ such that for any $x \in X$ with $\|x\| \geq \varepsilon$ and any weakly null sequence $\{x_n\}$ in the unit sphere of $X$, the inequality $1 + r \leq \liminf_{n \to \infty} \|x_n + x\|$ holds.

For a bounded set $A \subset X$, the ball-measure of noncompactness was defined by $\beta(A) = \inf \{\varepsilon > 0 : A\text{ can be covered by finitely many balls with diameter } \leq \varepsilon\}$. The function $\Delta$ defined by $\Delta(\varepsilon) = \inf \{1 - \inf \|x\| : x \in A\} : A\text{ is closed convex subset of } B(X)\text{ with } \beta(A) \leq \varepsilon\}$
is called the modulus of noncompact convexity. A Banach space $X$ is said to have property $L$, if $\lim_{\varepsilon \to 0} \Delta(\varepsilon) = 1$. This property is an important concept in the fixed point theory and a Banach space $X$ possesses property $L$ if and only if it is reflexive and has the uniform Opial property.

A Banach space $X$ is said to satisfy the weak fixed point property if every nonempty weakly compact convex subset $C$ and every nonexpansive mapping $T : C \to C$ (\begin{equation*}
\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C
\end{equation*}) have a fixed point, that is, there exists $x \in C$ such that $T(x) = x$.

Property $L$ and the fixed point property were also studied by Goebel and Kirk [8], Toledano et al. [26], Benavides [1], Benavides and Phothi [2]. A Banach space $X$ is said to satisfy the weak fixed point property if every nonempty weakly compact convex subset $C$ and every nonexpansive mapping $T : C \to C$ have a fixed point, that is, there exists $x \in C$ such that $T(x) = x$.

A Banach space $X$ is called uniformly convex (UC) if for each $0 > \varepsilon > 0$, there is $0 > \delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \varepsilon$ implies that $\|\frac{1}{2}(x + y)\| < 1 - \delta$.

A sequence $\{x_n\}$ is said to be $\varepsilon$-separated sequence for some $0 > \varepsilon > 0$ if $\text{sep}(x_n) = \inf \|x_n - x_m\| : n \neq m > \varepsilon$.

A Banach space $X$ is called nearly uniformly convex (NUC) if for each $0 > \varepsilon > 0$, there exists $0 > \delta > 0$ such that for each element $x \in B(X)$ and each sequence $\{x_n\}$ in $B(X)$ with $\text{sep}(x_n) > \varepsilon$, there is an index $k$ for which $\|\frac{x + x_k}{2}\| < 1 - \delta$.

For a real vector space $X$, a function $\rho : X \to [0, \infty]$ is called a modular if it satisfies the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for all scalar $\alpha$ with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular $\rho$ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular $\rho$ on $X$, the space $X_\rho = \{x \in X : \rho(x) < \infty \text{ for some } \sigma > 0\}$ is called a modular space. In general, the modular is not subadditive and thus it does not behave as a norm or a distance. But we can associate the modular with an $F$-norm. A functional
\[ \|x\| : X \to [0, \infty) \] defines an F-norm if and only if

(i) \[ \|x\| = 0 \iff x = 0, \]

(ii) \[ \|ax\| = \|x\| \] whenever \( |a| = 1, \)

(iii) \[ \|x + y\| \leq \|x\| + \|y\|, \]

(iv) If \( a_n \to a \) and \( \|x_n - x\| \to 0 \), then \( \|a_n x_n - ax\| \to 0. \)

F-norm defines a distance on \( X \) by \( d(x, y) = \|x - y\| \). If the linear metric space \( (X, d) \) is complete, then it is called an F-space. The modular space \( X_{\rho} \) can be equipped with the following F-norm:

\[ \|x\| = \inf \left\{ \alpha > 0 : \rho \left( \frac{x}{\alpha} \right) \leq \alpha \right\}. \]

If the modular \( \rho \) is convex, then the equality \( \|x\| = \inf \left\{ \alpha > 0 : \rho \left( \frac{x}{\alpha} \right) \leq 1 \right\} \) defines a norm which is called the Luxemburg norm.

A modular \( \rho \) is said to satisfy the \( \delta_2 \) condition if for any \( \varepsilon > 0 \), there exist constants \( K \geq 2, a > 0 \) such that \( \rho(2u) \leq K\rho(u) + \varepsilon \) for all \( u \in X_{\rho} \) with \( \rho(u) \leq a. \) If \( \rho \) provides the \( \delta_2 \) condition for any \( a > 0 \) with \( K \geq 2 \) dependent on \( a \), then \( \rho \) provides the strong \( \delta_2 \) condition (briefly \( \rho \in \delta_2^* \)).

Let us denote by \( \ell^0 \) the space of all real sequences. The Cesàro sequence spaces

\[ \text{Ces}_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left( \frac{1}{n^p} \sum_{i=1}^{n} |x_i| \right)^p < \infty, 1 \leq p < \infty \right\}, \]

and

\[ \text{Ces}_\infty = \left\{ x \in \ell^0 : \sup_{n} n^{-1} \sum_{i=1}^{n} |x_i| < \infty \right\}, \]

were introduced by Shiue [24]. Jagers [10] determined the Köthe duals of the sequence space \( \text{Ces}_p (1 < p < \infty) \). It can be shown that the inclusion \( \ell_p \subset \text{Ces}_p \) is strict for \( 1 < p < \infty \) although it does not hold for \( p = 1 \). Also, Suantai [25] defined the generalized Cesàro sequence space by

\[ \text{ces}(p) = \left\{ x \in \ell^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \right\}, \]

where \( \rho(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n^p} \sum_{i=1}^{n} |x_i| \right)^p \). If \( p = (p_n) \) is bounded, then

\[ \text{ces}(p) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left( \frac{1}{n^p} \sum_{i=1}^{n} |x_i| \right)^p < \infty \right\}. \]

The sequence space \( C(s, p) \) was defined by Bilgin [3] as follows:

\[ C(s, p) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left( \sum_{k=1}^{r} k^{-s} |x_k| \right)^p < \infty, s \geq 0 \right\}. \]
for \( p = (p_r) \) with \( \inf p_r > 0 \), where \( \sum \) denotes a sum over the ranges \( 2^r \leq k < 2^{r+1} \). The special case of \( C(s, p) \) for \( s = 0 \) is the space

\[
\text{Ces}(p) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left( 2^{-r} \sum_k k^{-r} |x_k| \right)^{p_r} < \infty \right\}
\]

which was introduced by Lim [14]. Also, the inclusion \( \text{Ces}(p) \subseteq C(s, p) \) holds. A paranorm on \( C(s, p) \) is given by

\[
\rho(x) = \left( \sum_{r=0}^{\infty} \left( 2^{-r} \sum_k k^{-r} |x_k| \right)^{p_r} \right)^{1/M}
\]

for \( M = \max(1, H) \) and \( H = \sup p_r < \infty \).

The \( Z \)–transform of a sequence \( x = (x_k) \) is defined by \( (Zx)_n = y_n = \alpha x_n + (1-\alpha) x_{n-1} \) by using the Zweier operator \( Z = (z_{nk}) = \begin{cases} \alpha, & k = n \\ 1-\alpha, & k = n+1 \end{cases} \) for \( n, k \in \mathbb{N} \) and \( \alpha \in \mathbb{F} \setminus \{0\} \),

where \( \mathbb{F} \) is the field of all complex or real numbers. The Zweier operator was studied by Şengönül and Kayaduman [23].

Now we introduce a new modular sequence space \( \mathbb{Z}_\sigma(s, p) \) by

\[
\mathbb{Z}_\sigma(s, p) = \left\{ x \in \ell^0 : \sigma(x) < \infty, \text{ for some } t > 0 \right\},
\]

where \( \sigma(x) = \sum_{r=0}^{\infty} \left( 2^{-r} \sum_k k^{-r} |x_k| \right)^{p_r} s^r < \infty \) and \( s \geq 0 \). If we take \( \alpha = 1 \), then \( \mathbb{Z}_\sigma(s, p) = C(s, p) \); if \( \alpha = 1 \) and \( s = 0 \), then \( \mathbb{Z}_\sigma(s, p) = \text{Ces}(p) \). It can be easily seen that \( \sigma : \mathbb{Z}_\sigma(s, p) \to [0, \infty] \) is a modular on \( \mathbb{Z}_\sigma(s, p) \). We define the Luxemburg norm on the sequence space \( \mathbb{Z}_\sigma(s, p) \) as follows:

\[
\|x\|_\sigma = \inf \left\{ t > 0 : \sigma\left( \frac{x}{t} \right) \leq 1 \right\}, \quad \forall x \in \mathbb{Z}_\sigma(s, p).
\]

It is easy to see that the space \( \mathbb{Z}_\sigma(s, p) \) is a Banach space with respect to the Luxemburg norm.

Throughout the paper, suppose that \( p = (p_r) \) is bounded with \( p_r > 1 \) for all \( r \in \mathbb{N} \) and

\[
e_i = \left( 0, 0, \ldots, 0, 1, 0, 0, 0, \ldots \right)
\]

\[
x_i = (x(1), x(2), x(3), \ldots, x(i), 0, 0, 0, \ldots),
\]

\[
x_{i-1} = (0, 0, 0, \ldots, x(i+1), x(i+2), \ldots),
\]

for \( i \in \mathbb{N} \) and \( x \in \ell^0 \). In addition, we will require the following inequalities:

\[
|a_k + b_k|^p \leq C \left( |a_k|^p + |b_k|^p \right),
\]

\[
|a_k + b_k|^p \leq |a_k|^p + |b_k|^p,
\]
where \( t_k = \frac{p_k}{M} \leq 1 \) and \( C = \max\{1, 2^{H-1}\} \) with \( H = \sup p_k \).

3. Main results

Since \( \ell_p \) is reflexive and convex, \( \ell(p) \) - type spaces have many useful applications, and it is natural to consider a geometric structure of these spaces. From this point of view, we generalized the space \( C(s, p) \) by using the Zweier operator and then obtained the equality \( \mathcal{Z}_\sigma(s, p) = \text{Ces}(p) \), that is, it was seen that the structure of the space \( \text{Ces}(p) \) was preserved. In this section, our goal is to investigate a geometric structure of the modular space \( \mathcal{Z}_\sigma(s, p) \) related to the fixed point theory. For this, we will examine property \( (\beta) \) and the uniform Opial property for \( \mathcal{Z}_\sigma(s, p) \). Finally, we will give some fixed point results. To do this, we need some results which are important in our opinion.

**Lemma 3.1** [6] If \( \sigma \in \delta^2_\sigma \), then for any \( L > 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
|\sigma(u + v) - \sigma(u)| < \epsilon,
\]
where \( u, v \in X_\sigma \) with \( \sigma(u) \leq L \) and \( \sigma(v) \leq \delta \).

**Lemma 3.2** [6] If \( \sigma \in \delta^2_\sigma \), convergence in norm and in modular are equivalent in \( X_\sigma \).

**Lemma 3.3** [6] If \( \sigma \in \delta^2_\sigma \), then for any \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon) > 0 \) such that \( \|x\| \geq 1 + \delta \) implies \( \sigma(x) \geq 1 + \epsilon \).

Now we give the following two lemmas without proof.

**Lemma 3.4** If \( \|x\| \leq 1 \) for any \( x \in \mathcal{Z}_\sigma(s, p) \), then \( \sigma(x) \leq \|x\| \).

**Lemma 3.5** For any \( x \in \mathcal{Z}_\sigma(s, p) \), \( \|x\| = 1 \) if and only if \( \sigma(x) = 1 \).

**Lemma 3.6** If \( \liminf p_r > 1 \), then for any \( x \in \mathcal{Z}_\sigma(s, p) \), there exist \( k_0 \in \mathbb{N} \) and \( \mu \in (0, 1) \) such that
\[
\sigma\left(\frac{x^k}{2}\right) \leq 1 - \frac{\mu}{2} \sigma(x^k)
\]
for all \( k \in \mathbb{N} \) with \( k \geq k_0 \), where \( x^k = \left(\frac{1}{k-1}, 0, \ldots, 0, \sum_{2^i \leq k} x(i), x(k+1), x(k+2), \ldots \right) \) and \( 2^k \leq k < 2^{k+1} \).

**Proof** Let \( k \in \mathbb{N} \) be fixed. Then there exists \( n_k \in \mathbb{N} \) such that \( k \in I_{n_k} \). Let \( \gamma \) be a real number \( 1 < \gamma \leq \liminf p_r \), and so there exists \( k_0 \in \mathbb{N} \) such that \( \gamma < p_{k_0} \) for all \( k \geq k_0 \).

Choose \( \mu \in (0, 1) \) such that \( \left(\frac{1}{2}\right)^\gamma \leq 1 - \frac{\mu}{2} \). Therefore, we have
\[ \sigma \left( \frac{x^k}{2} \right) = \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{r} k^{-r} \left[ \frac{\alpha x(k) + (1-\alpha) x(k-1)}{2} \right]^{p_r} \right) \]

\[ = \sum_{r=0}^{\infty} \left( \frac{1}{2} \right)^{p_r} \left( 2^{-r} \sum_{r} k^{-r} \left[ \frac{\alpha x(k) + (1-\alpha) x(k-1)}{2} \right]^{p_r} \right) \]

\[ \leq \left( \frac{1}{2} \right)^{r} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{r} k^{-r} \left[ \frac{\alpha x(k) + (1-\alpha) x(k-1)}{2} \right]^{p_r} \right) \]

\[ < \frac{1-\mu}{2} \sigma(x^k) \]

for each \( x \in \mathbb{Z}_\sigma(s, p) \) and \( k \geq k_0 \).

**Lemma 3.7** If \( \sigma \in \delta'_2 \), then for any \( \epsilon \in (0,1) \), there exists \( \delta \in (0,1) \) such that \( \sigma(x) \leq 1-\epsilon \) implies \( \|x\| \leq 1-\delta \).

**Proof** Suppose that lemma does not hold. So, there exist \( \epsilon > 0 \) and \( x_n \in \mathbb{Z}_\sigma(s, p) \) such that \( \sigma(x_n) < 1-\epsilon \) and \( \frac{1}{2} \leq \|x_n\| \to 1 \). Take \( s_n = \frac{1}{\|x_n\| - 1} \), and so \( s_n \to 0 \) as \( n \to \infty \). Let \( P = \sup \{ \sigma(2x_n) : n \in \mathbb{N} \} \). There exists \( D \geq 2 \) such that \( \sigma(2u) \leq D \sigma(u) + 1 \) for every \( u \in \mathbb{Z}_\sigma(s, p) \) with \( \sigma(u) < 1 \), since \( \sigma \in \delta'_2 \). We have

\[ \sigma(2x_n) \leq D \sigma(x_n) + 1 < D + 1 \]

for all \( n \in \mathbb{N} \) by (3.1). Therefore, \( 0 < P < \infty \) and from Lemma 3.5 we have

\[ 1 = \sigma \left( \frac{x_n}{\|x_n\|} \right) = \sigma(2s_n x_n + (1-s_n)x_n) \]

\[ \leq s_n \sigma(2x_n) + (1-s_n) \sigma(x_n) \]

\[ \leq s_n P + (1-\epsilon) \to (1-\epsilon). \]

This is a contradiction. So, the proof is complete.

**Theorem 3.8** The space \( \mathbb{Z}_\sigma(s, p) \) has property \( (\beta) \).

**Proof** Let \( \epsilon > 0 \) and \( (x_n) \subset B(\mathbb{Z}_\sigma(s, p)) \) with \( \text{sep}(x_n) \geq \epsilon \) and \( x \in B(\mathbb{Z}_\sigma(s, p)) \). For each \( l \in \mathbb{N} \), we can find \( r_l \in \mathbb{N} \) such that \( 2^{r_l} \leq l < 2^{r_{l+1}} \). Let

\[ x_n^l = \left\{ 0,0,...,0, \sum_{x \geq l} |x(i)|, x_n(l+1), x_n(l+2),... \right\} \]

Since for each \( i \in \mathbb{N} \), \( (x_n(i))_{\geq l} \) is bounded, by using the diagonal method, we can find a subsequence \( (x_n) \) of \( (x_n) \) such that \( (x_n(i)) \) converges for each \( i \in \mathbb{N} \) with \( 1 \leq i \leq l \). Therefore, there exists an increasing sequence of positive integers \( t_i \) such that \( \text{sep}(x_n^{t_i}) \geq \epsilon \). Thus, there exists a sequence of positive integers \( (r_l)_{\geq l} \) with \( r_1 < r_2 < ... \)
such that $\|x_i\| \geq \frac{\varepsilon}{2}$ for all $i \in \mathbb{N}$. Since $\sigma \in \mathcal{D}_2$, there is $\eta > 0$ such that

$$\sigma(x_i) \geq \eta$$

for all $i \in \mathbb{N}$ from Lemma 3.3. However, there exist $k_0 \in \mathbb{N}$ and $\mu \in (0,1)$ such that

$$\sigma\left(\frac{v}{2}\right) \leq \frac{1-\mu}{2} \sigma(v)$$

for all $v \in \mathcal{Z}_\sigma(s, p)$ and $k \geq k_0$ by Lemma 3.6. There exists $\delta > 0$ such that

$$\sigma(y) \leq 1 - \frac{\mu \eta}{4} \Rightarrow \|y\| \leq 1 - \delta$$

for any $y \in \mathcal{Z}_\sigma(s, p)$ by Lemma 3.7.

By Lemma 3.1, there exists $\delta_0$ such that

$$|\sigma(u + v) - \sigma(u)| < \frac{\mu \eta}{4},$$

where $\sigma(u) \leq 1$ and $\sigma(v) \leq \delta_0$. Hence, we get that $\sigma(x) \leq 1$ since $x \in B(\mathcal{Z}_\sigma(x, p))$. Then there exists $k \geq k_0$ such that $\sigma(x^i) \leq \delta_0$. Let $u = x_i$ and $v = x^i$. Then

$$\sigma\left(\frac{u}{2}\right) < 1 \text{ and } \sigma\left(\frac{v}{2}\right) < \delta_0.$$ 

We obtain from (3.3) and (3.5) that

$$\sigma\left(\frac{u + v}{2}\right) \leq \sigma\left(\frac{u}{2}\right) + \frac{\mu \eta}{4} \leq \frac{1-\mu}{2} \sigma(u) + \frac{\mu \eta}{4}.$$

Choose $s_i = r_i$. By the inequalities (3.2), (3.3), (3.6) and the convexity of the function $f(u) = \|u\|^{p_r}$, we have

$$\sigma\left(\frac{x + x_i}{2}\right) = \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha(x(k) + x_i(k)) + (1-\alpha)(x(k-1) + x_i(k-1))\right|\right)^{p_r}$$

$$= \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha(x(k) + x_i(k)) + (1-\alpha)(x(k-1) + x_i(k-1))\right|\right)^{p_r}$$

$$+ \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha(x(k) + x_i(k)) + (1-\alpha)(x(k-1) + x_i(k-1))\right|\right)^{p_r}$$

$$\leq \frac{1}{2} \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha x(k) + (1-\alpha)x(k-1)\right|\right)^{p_r}$$

$$+ \frac{1}{2} \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha x_i(k) + (1-\alpha)x_i(k-1)\right|\right)^{p_r}$$

$$+ \sum_{\alpha=0}^{n-1} \left(2^{-r} \sum_{r} k^{-r} \left|\alpha x(k) + (1-\alpha)x(k-1)\right|\right)^{p_r} + \frac{\mu \eta}{4}.$$
\[
\begin{align*}
\leq & \frac{1}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} |\alpha(k) + (1 - \alpha)x(k-1)|^p \right) \\
& + \frac{1}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} |\alpha_x(k) + (1 - \alpha)x(k-1)|^p \right) \\
& + \frac{1}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} \left| \frac{\alpha_x(k) + (1 - \alpha)x(k-1)}{2} \right|^p \right) + \frac{\mu \eta}{4} \\
\leq & \frac{1}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} |\alpha(k) + (1 - \alpha)x(k-1)|^p \right) \\
& + \frac{1}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} |\alpha_x(k) + (1 - \alpha)x(k-1)|^p \right) \\
& - \frac{\mu}{2} \sum_{r=0}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} \left| \frac{\alpha_x(k) + (1 - \alpha)x(k-1)}{2} \right|^p \right) + \frac{\mu \eta}{4} \\
& \leq 1 - \frac{\mu \eta}{2} + \frac{\mu \eta}{4} \\
& = 1 - \frac{\mu \eta}{4}.
\end{align*}
\]

So, the inequality (3.4) implies that \(\left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta\). Consequently, the space \(Z_\sigma(s, p)\) possesses property (\(\beta\)).

Since property (\(\beta\)) implies NUC, NUC implies property (\(D\)) and property (\(D\)) implies reflexivity, we can give the following result from Theorem 3.8.

**Corollary 3.9** The space \(Z_\sigma(s, p)\) is nearly uniform convex, reflexive and also it has property (\(D\)).

**Theorem 3.10** The space \(Z_\sigma(s, p)\) has the uniform Opial property.

**Proof** Let \(\varepsilon > 0\) and \(x \in Z_\sigma(s, p)\) be such that \(\|x\| \geq \varepsilon\) and \((x_n)\) be a weakly null sequence in \(S(Z_\sigma(s, p))\). By \(\sigma \in \mathcal{D}'_1\), there exists \(\zeta \in (0,1)\) independent of \(x\) such that \(\sigma(x) > \zeta\) by Lemma 3.2. Also since \(\sigma \in \mathcal{D}'_2\), by Lemma 3.1, there is \(\zeta_1 \in (0, \zeta)\) such that

\[
|\sigma(y + z) - \sigma(y)| < \frac{\zeta_1}{4}
\]

whenever \(\sigma(y) \leq 1\) and \(\sigma(z) \leq \zeta_1\). Take \(r_0 \in \mathbb{N}\) such that

\[
\sum_{r=r_0+1}^{\infty} \left( 2^{-r} \sum_{k=0}^{\infty} k^{-r} |\alpha(k) + (1 - \alpha)x(k-1)|^p \right) < \frac{\zeta_1}{4}.
\]

Hence, we have
\[ \zeta < \sum_{r=1}^{g_0} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(k) + (1 - \alpha) x(k - 1)| \right)^{p_r} + \sum_{r=g_0+1}^{g} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(k) + (1 - \alpha) x(k - 1)| \right)^{p_r} \]
\[ \leq \sum_{r=1}^{g_0} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(k) + (1 - \alpha) x(k - 1)| \right)^{p_r} + \frac{\zeta}{4} \]
and this implies that
\[ \sum_{r=1}^{g_0} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(k) + (1 - \alpha) x(k - 1)| \right)^{p_r} > \zeta - \frac{\zeta}{4} \]
\[ > \zeta - \frac{\zeta}{4} = \frac{3\zeta}{4}. \]

Since \( x_n \overset{w}{\rightarrow} 0 \), by the inequality (3.10), there exists \( r_0 \in \mathbb{N} \) such that
\[ \sum_{r=r_0}^{g_0} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(x_n(k) + x(k)) + (1 - \alpha)(x_n(k-1) + x(k-1))| \right)^{p_r} > \frac{3\zeta}{4}. \]
Again, by \( x_n \overset{w}{\rightarrow} 0 \), there is \( r_1 > r_0 \) such that for all \( r > r_1 \)
\[ \|x_{n_k}\| < 1 - \left[ 1 - \frac{\zeta}{4} \right]^{1/M}, \]
where \( p_r \leq M \in \mathbb{N} \) for all \( r \in \mathbb{N} \). Therefore, we obtain that
\[ \|x_{n_k}\| > \left[ 1 - \frac{\zeta}{4} \right]^{1/M} \]
by the triangle inequality of the norm. It follows from the definition of the Luxemburg norm that
\[ 1 \leq \sigma \left( \frac{x_{n_k}}{\left( 1 - \frac{\zeta}{4} \right)^{1/M}} \right) \]
\[ = \sum_{r=r_1+1}^{\infty} \left( \frac{2^{-r} \sum_{k} k^{-r} |\alpha(x_n(k) + x(k))|}{\left( 1 - \frac{\zeta}{4} \right)^{1/M}} \right)^{p_r} \]
\[ \leq \left( \frac{1}{\left( 1 - \frac{\zeta}{4} \right)^{1/M}} \right)^M \sum_{r=r_1+1}^{g} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(x_n(k) + (1 - \alpha)x_n(k-1))| \right)^{p_r} \]
and this implies that
\[ \sum_{r=r_1+1}^{\infty} \left( 2^{-r} \sum_{k} k^{-r} |\alpha(x_n(k) + (1 - \alpha)x_n(k-1))| \right)^{p_r} \geq 1 - \frac{\zeta}{4}. \]
By (3.7), (3.8), (3.11), (3.15) and since \( x_n \overset{w}{\rightarrow} 0 \Rightarrow x_n \overset{coordinatewise}{\rightarrow} 0 \), we have for any
\[ r > r_1 \] that
\[
\sigma(x_n + x) = \sum_{r=1}^{n} \left( 2^{-r} \sum_{r} k^{-r} |\alpha(x_n(k) + x(k)) + (1 - \alpha)(x_n(k-1) + x(k-1)) | \right)^
u
\]
\[
+ \sum_{r=n+1}^{\infty} \left( 2^{-r} \sum_{r} k^{-r} |\alpha(x_n(k) + x(k)) + (1 - \alpha)(x_n(k-1) + x(k-1)) | \right)^
u
\]
\[
\geq \sum_{r=n+1}^{\infty} \left( 2^{-r} \sum_{r} k^{-r} |\alpha(x_n(k) + x(k)) + (1 - \alpha)(x_n(k-1) + x(k-1)) | \right)^
u
\]
\[
- \frac{\zeta}{4} + \frac{3\zeta}{4}
\]
\[
\geq \frac{3\zeta}{4} + \left( 1 - \frac{\zeta}{4} \right) - \frac{\zeta}{4}
\]
\[
= 1 + \frac{\zeta}{4}
\]

Since \( \sigma \in \delta_*^2 \), it follows from Lemma 3.3 that there is \( \tau \) depending on \( \zeta \) only such that \[ \|x_n + x\| \geq 1 + \tau. \]

**Corollary 3.11** The space \( Z_n(s, p) \) has property \( (L) \) and the fixed point property.

**REFERENCES**


